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Finite 2-Groups with Small Centralizer of an Involution

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1. INTRODUCTION

The starting point is the following theorem of Berkovich [2]. For a finite p-group G, one of the following holds:

(a) G has no maximal elementary abelian subgroup of order p^2.

(b) $|\Omega_1(G)| \leq p^2$.

(c) There exists in G an element x of order p such that $C_G(x) = \langle x \rangle \times Q$, where Q is a cyclic or generalized quaternion. Furthermore, G has no normal subgroup of order p^{p+1} and exponent p.

The classification of p-groups containing an element t of order p such that $C_G(t) = \langle t \rangle \times Z$, where Z is cyclic of order p^m, is very difficult. This question is solved for $m = 1$ by Suzuki (see [5, Satz 14.23]). In this case we have a well-known characterization of p-groups of maximal class. For $p = 2$ and $m = 2$, the problem is solved in Gorenstein [4, Proposition 10.27]. For $p > 2$ there are some results of Blackburn [3] which only show that the problem is a difficult one indeed. For $p = 2$, the problem is solved in a special case (with two other assumptions) in Berkovich [2, Theorem 9.2].

In this paper we consider the case $p = 2$ in general. In fact we shall prove the following classification result and in the case where G has elementary abelian subgroups of order 8, we get exactly four infinite classes of 2-groups which we give in terms of generators and relations.

THEOREM 1.1. *Let G be a finite nonabelian 2-group containing an involution t such that the centralizer $C_G(t) = \langle t \rangle \times C$, where C is a cyclic group*

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of order 2^m, $m \geq 1$. *Then* G *has no elementary abelian subgroup of order* 16 *and* G *is generated by at most three elements*.

(A1) *If* G *has no elementary abelian subgroup of order* 8, *then one of the following assertions holds*:

(a) G *is a dihedral group* D_{2^n} *of order* 2^n $(n \geq 3)$ *or* G *is a semidihedral group* SD_{2^n} *of order* 2^n $(n \geq 4)$. *Here we have* $m = 1$.

(b) G *is isomorphic to the group* M_{2^n} *of order* 2^n $(n \geq 4)$ *and this is the unique* 2-*group of class* 2 *and order* 2^n *which has a cyclic subgroup of index* 2. *Here we have* $m = n - 2$.

(c) $|G : C_G(t)| = 2$, $t \in \Phi(G)$, $Z(G)$ *is a cyclic subgroup of order* ≥ 4 *not contained in* $\Phi(C_G(t))$, $G/Z(G)$ *is dihedral with the cyclic subgroup* $T/Z(G)$ *of index* $2, T$ *is abelian of type* $(2, 2^m), m \geq 3$ *and* $t \in T$. *If* x *is an element of maximal order in* $G \backslash T$, *then* $\langle x^2 \rangle = Z(G)$. *Here we have* $C_G(t) = T$ *and* $|G| = 2^{m+2}$.

(d) G *has a subgroup* S *of index* ≤ 2, *where* $S = A\,L$, *the subgroup* L *is normal in* G, $L = \langle b, t \mid b^{2^{n-1}} = 1,\ t^2 = 1,\ b^t = b^{-1},\ n \geq 3 \rangle \cong D_{2^n}$, $A = \langle a \rangle$ *is cyclic order* 2^m, $m \geq 2$, $A \cap L = Z(L)$, $[a, t] = 1$, $\Omega_1(G) = \Omega_1(S) = \Omega_2(A) * L$ *which is the central product of* $\Omega_2(A)$ *and* L, *where* $\Omega_2(A) \cap L = Z(L)$. *If* $|G : S| = 2$, *then there is an element* $x \in G \setminus S$ *so that* $t^x = t\,b$ *and* $C_G(t) = \langle t \rangle \times \langle a \rangle$.

(A2) *If* G *has an elementary abelian subgroup of order* 8, *then* $Z(G)$ *is of order* 2, $C_G(t) = \langle t \rangle \times \langle a \rangle$, *where* $A = \langle a \rangle$ *has order* 2^m $(m \geq 2)$, G *has a normal subgroup* $L = \langle b, t \mid b^{2^{n-1}} = 1,\ n \geq 3,\ t^2 = 1,\ b^t = b^{-1} \rangle \cong D_{2^n}$, $A \cap L = Z(L) = \langle z \rangle$, $S = A\,L$ *is a normal subgroup of index* ≤ 2 *in* G, *and* G *is isomorphic to one of the following groups*:

(e) $G = \langle a, b, t \mid a^{2^m} = 1,\ m \geq 3,\ b^{2^{n-1}} = 1,\ n \geq 4,\ t^2 = 1,\ b^t = b^{-1},\ [a, t] = 1,\ a^{2^{m-1}} = b^{2^{n-2}} = z,\ b^a = b^{1+2^i},\ i = n - m \geq 2 \rangle$. *We have* $G = A\,L = S$ *and the cyclic group* $\langle a \rangle / \langle z \rangle$ *of order* 2^{m-1} *acts faithfully on* L.

(*f*) $G = \langle a, b, t, s \mid a^{2^m} = 1,\ a^{2^{m-1}} = z,\ m \geq 4,\ b^{2^{n-1}} = 1,\ n \geq 5,\ b^{2^{n-2}} = z,\ t^2 = 1,\ b^t = b^{-1},\ [t, a] = 1,\ b^a = b^{1+2^i},\ i = n - m + 1,\ s^2 = 1,\ b^s = b^{-1},\ t^s = t\,b,\ s^a = a^{2^{m-2}}\, b^{-2^{i-1}} s,\ i \geq 2 \rangle$. *Here* $S = A\,L$ *is a subgroup of index* 2 *in* G *and* $\Omega_1(S) = \Omega_2(A) * L$. *Also* $G = S\langle s \rangle$, *where* $M = L\langle s \rangle \cong D_{2^{n+1}}$, $N_G(M) = M\langle a^2 \rangle$ *and* s *acts invertingly on* $\Omega_2(A)$. *We have* $|G : S| = 2$ *and so the order of* G *is* 2^{m+n}.

(*g*) *Berkovich groups* $G = F(m, n)$ *from* [2, *Theorem* 9.2]. *Here we have* $G = \langle a, b, t, s \mid a^{2^m} = 1,\ a^{2^{m-1}} = z,\ m \geq 2,\ b^{2^{n-1}} = 1,\ n \geq 3,\ b^{2^{n-2}} = z,\ t^2 = 1,\ b^t = b^{-1},\ [t, a] = 1,\ [b, a] = 1,\ s^2 = 1,\ b^s = b^{-1},\ t^s = tb,\ a^s = a^{-1} z^v,\ v = 0, 1$, *and if* $v = 1$, *then* $m \geq 3 \rangle$. *Here* $S = A * L$ *is the central product of* A *and* L *and* $G = S\langle s \rangle$, *where* $|G : S| = 2$. *We have* $M = L\langle s \rangle \cong D_{2^{n+1}}$ *and* s *acts invertingly on* $\langle a \rangle$ *or in case* $m \geq 3$ *we can have also* $a^s = a^{-1} z$.

(h) $G = \langle a, b, t, s \mid a^{2^m} = 1,\ m \geq 4,\ b^{2^{n-1}} = 1,\ n \geq 4,\ a^{2^{m-1}} = b^{2^{n-2}} = z,\ t^2 = 1,\ b^t = b^{-1},\ s^2 = 1,\ b^s = b^{-1},\ t^s = tb,\ [a, t] = 1,\ b^a = bz,\ a^s = a^{-1+2^{m-2}} b^{2^{n-3}} \rangle$. *Here we have again* $G = S\langle s \rangle$ *so that* $|G : S| = 2$. *Finally,* $M = L\langle s \rangle$ *is isomorphic to* $D_{2^{n+1}}$.

2. NOTATION AND KNOWN RESULTS

Let C_m be the cyclic group of order m, E_{p^m} the elementary abelian group of order p^m (p prime), D_{2^m} the dihedral group of order 2^m ($m \geq 3$), SD_{2^m} the semidihedral group of order 2^m ($m \geq 4$), Q_{2^m} the generalized quaternion group of order 2^m ($m \geq 3$),

$$M_{2^m} = \langle x, y \mid x^2 = y^{2^{m-1}} = 1, m \geq 4, [y, x] = y^{2^{m-2}} \rangle,$$

$C_G(M)$ the centralizer of a subset M in G, $N_G(H)$ the normalizer of a subgroup H in G, G' the derived group of G, $Z(G)$ the center of G, $\Phi(G)$ the Frattini subgroup of G, and for a p-group G (p is a prime) we set $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1 \rangle$ and $\mho_n(G) = \langle x^{p^n} \mid x \in G \rangle$.

A p-group G of order p^m is said to be of maximal class if$m \geq 3$ and the class of G is $m - 1$. For $x \in G$, we denote by $ccl_G(x)$ the conjugacy class of x in G. Two elements x, y in G are said to be fused in G if there is $g \in G$ such that $x^g = g^{-1}xg = y$. Finally, $[x, y] = x^{-1}y^{-1}xy$ is the commutator of x and y. We shall use freely the following known results:

PROPOSITION 2.1 (Berkovich [1, Proposition 19(b)]). *Let B be a subgroup of a nonabelian p-group G such that $C_G(B) \subseteq B$. If $|B| = p^2$, then G is of maximal class.*

PROPOSITION 2.2 (Huppert [5, p. 90]). *Let G be a nonabelian 2-group which possesses a cyclic subgroup of index 2. Then G is isomorphic to one of the following groups*: $D_{2^n} (n \geq 3)$, $SD_{2^n} (n \geq 4)$, $Q_{2^n} (n \geq 3)$, *or* $M_{2^n} (n \geq 4)$.

PROPOSITION 2.3 (Berkovich [2, Theorem 10.3]). *Let G be a nonabelian 2-group. If $|G : G'| = 4$, then G has a cyclic subgroup of index 2. In particular, if G is a 2-group of maximal class, then G is isomorphic to D_{2^n}, SD_{2^n}, or Q_{2^n}.*

PROPOSITION 2.4 (Huppert [5, p. 304]). *Let G be a p-group in which every normal abelian subgroup is cyclic. Then G is either cyclic or a 2-group of maximal class.*

PROPOSITION 2.5 (Huppert [5, p. 84]). *Let $G = \langle b \rangle$ be a cyclic group of order 2^n ($n \geq 3$). Then the automorphism group* Aut(G) *is abelian of type $(1, n-2)$ and $G = \langle \alpha \rangle \times \langle \beta \rangle$, where α is induced by $b^\alpha = b^{-1}$ and β is induced by $b^\beta = b^5 = b^{1+2^2}$. Here α is of order 2 and β is of order 2^{n-2}. We have $C_G(\alpha) = \langle b^{2^{n-1}} \rangle$, which is the fixed subgroup of α in G (of order 2). The* *fixed subgroup of β in G is $\langle b^{2^{n-2}}\rangle$ which is of order* 4. *The fixed subgroup of β^{2^j} in G has order 2^{2+j} $(0 \le j \le n-2)$. On the other hand, the automorphism γ induced by $b^\gamma = b^{1+2^i}(2 \le i \le n)$ has the fixed subgroup of order 2^i. Hence there is an odd number r so that*

$$b^{\beta^{r2^{i-2}}} = b^{1+2^i} \qquad (2 \le i \le n).$$

PROPOSITION 2.6. *The automorphism group* $\mathrm{Aut}(G)$ *of* $G = \langle b, t \mid b^{2^n} = 1, t^2 = 1, b^t = b^{-1}\rangle = D_{2^{n+1}}$ $(n \ge 3)$ *is generated by the inner automorphism group* $\mathrm{Inn}(G)$ (*which is isomorphic to* D_{2^n}) *and "outer" automorphisms α and β, where α is of order* 2 *and is induced with $t^\alpha = tb$, $b^\alpha = b^{-1}$, and β is of order 2^{n-2} and is induced with $t^\beta = t$, $b^\beta = b^5$. We have $[\alpha, \beta] = i_{b^2}$, which is the inner automorphism of G induced* (*by conjugation*) *with the element b^2. Hence the outer automorphism group* $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ *is abelian of type* $(1, n-2)$. *Furthermore* $C_G(\beta^{2^j}) \cong D_{2^{j+3}}$ $(0 \le j \le n-2)$.

Proof. The inner automorphisms of G induced with elements contained in $\langle b\rangle$ fuse the 2^n involutions in $G \setminus Z(G)$ in two orbits $O_1 = t\langle b^2\rangle$ and $O_2 = tb\langle b^2\rangle$ of lengths 2^{n-1} each. The inner automorphism i_t (induced by t) fixes t and acts invertingly on $\langle b\rangle$. Consider the outer automorphism α of order 2 induced by $t^\alpha = tb$, $b^\alpha = b^{-1}$ so that α fuses O_1 and O_2. Let $B \supseteq \mathrm{Inn}(G)$ be the subgroup of $\mathrm{Aut}(G)$ which fixes O_1 (and O_2) so that $|\mathrm{Aut}(G) : B| = 2$, B is normal in $\mathrm{Aut}(G)$, and $\mathrm{Aut}(G) = \langle\alpha\rangle B$ which is a semidirect product of $\langle\alpha\rangle$ and B. Let β'' be any outer automorphism from $B \setminus \mathrm{Inn}(G)$. Then multiplying β'' with an inner automorphism i induced with an element contained in $\langle b\rangle$, we may assume that $\beta' = \beta'' i$ fixes the involution t. But $\langle\beta'\rangle$ must act faihfully on the (characteristic) cyclic subgroup $\langle b\rangle$ of G. Multiplying β' with i_t (if necessary), we may assume that $\beta_0 = \beta'' i\, i_t^\epsilon$ $(\epsilon = 0, 1)$ fixes t and centralizes a subgroup of order ≥ 4 in $\langle b\rangle$. By Proposition 2.5, we see that β_0 acts on $\langle b\rangle$ as a power of the automorphism β from Proposition 2.5. Hence if we consider the outer automorphism β of order 2^{n-2} induced by $t^\beta = t$, $b^\beta = b^5$, we see that B is a semidirect product of $\langle\beta\rangle$ and $\mathrm{Inn}(G)$ and so $\mathrm{Aut}(G) = \mathrm{Inn}(G)\langle\alpha, \beta\rangle$. Finally, we compute that $[\alpha, \beta] = i_{b^2}$ and so $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is abelian of type $(1, n-2)$. It follows from Proposition 2.5 that the fixed subgroup of β^{2^j} in G is isomorphic to $D_{2^{j+3}}$ $(0 \le j \le n-2)$. ∎

3. PROOF OF THEOREM 1.1

Let G be a nonabelian 2-group containing an involution t such that $C_G(t) = \langle t\rangle \times C$, where $C \cong C_{2^m}$, $m \ge 1$. If $m = 1$, then by Propositions 2.1 and 2.3, we have that G is isomorphic to D_{2^n}, $n \ge 3$, or SD_{2^n}, $n \ge 4$. In what follows we assume that $m \ge 2$. Since G is not of maximal class, it follows by

Proposition 2.4 that G possesses a normal four-subgroup U. Set $T = C_G(U)$ so that we have $|G : T| \le 2$. Suppose in addition that $t \in T$. This forces $t \in U$ since $C_G(t)$ does not have an elementary abelian subgroup of order 8. Also we have $|G : T| = 2$ and $T = \langle t\rangle \times C$, where $C \cong C_{2^m}$, $m \ge 2$. We may also assume that G does not possess any other normal four-subgroup which possibly does not contain t because we shall consider that case later. Set $\Omega_1(C) = \langle u\rangle = \mho_{m-1}(T)$ so that $U = \langle t, u\rangle = \Omega_1(T)$ and $\langle u\rangle \subseteq Z(G)$. Since $Z(G) \subset T$, so $\langle u\rangle = \Omega_1(Z(G))$. We have $\Phi(G) = \mho_1(G) \subset T$ and $\Phi(G)$ contains $\mho_1(C)$ which is of order 2^{m-1}. If t is not in $\Phi(G)$, then there exists a maximal subgroup M of G which does not contain t. Set $M_0 = T \cap M$ so that $M_0 \cong C_{2^m}$ is a cyclic subgroup of index 2 in M and $C_G(t) = \langle t\rangle \times M_0$. If M is cyclic, then t acts faithfully on M and t centralizes the maximal subgroup M_0 of M which gives that G is isomorphic to $M_{2^{m+2}}$, $m \ge 2$, and this is case (b) of Theorem 1.1. So assume now that M is not cyclic but M has the cyclic subgroup M_0 of index 2. If M were abelian, then $\Omega_1(M)$ is a normal four-subgroup of G which does not contain t, contrary to our assumption. Thus M is nonabelian. If M is not of maximal class, then again $\Omega_1(M)$ is a normal four-subgroup of G which does not contain t. Hence M is a group of maximal class and we have $Z(M) = \langle u\rangle$, where $\langle t, u\rangle = U$. Let $\langle v\rangle$ be the cyclic subgroup of order 4 contained in M_0 and let $y \in M \setminus M_0$ so that we have $v^y = v^{-1} = vu$ and $t^y = tu$. Hence we have $(tv)^y = tv$, where $\langle tv\rangle$ is a cyclic subgroup of order 4 in $T \setminus M_0$ with $(tv)^2 = u$ and $C_G(tv) \supseteq \langle T, y\rangle = G$. It follows that G is the central product of a cyclic group of order 4 and a 2-group of maximal class. But then in any case G contains a dihedral subgroup of order 2^{m+1}, $m \ge 2$, and the center of G is cyclic of order 4. This group is then a special case of groups in (d) of Theorem 1.1 (with $S = G$ and A is of order 4 centralising L). It remains to consider the case $t \in \Phi(G)$. In this case we have $\Phi(G) = \langle t\rangle \times \mho_1(C)$ and so the group G is generated by two elements. If $m = 2$, then $\Phi(G) = \langle t, u\rangle$. Since $\mho_1(T) = \langle u\rangle$ there is an element x in $G \setminus T$ so that $x^2 \in U \setminus \langle u\rangle$. But then x centralizes U, which is a contradiction. Hence we must have $m \ge 3$. There is an element $x \in G \setminus T$ such that $x^2 \in \Phi(G) \setminus \mho_1(C) \setminus U$. Set $C = \langle a\rangle$ which is of order 2^m, $m \ge 3$, so that $a^{2^{m-1}} = u$, $T = C_G(t) = \langle t\rangle \times \langle a\rangle$, and $U = \Omega_1(T) = \langle t, u\rangle$. Also, $\mho_1(T) = \mho_1(C) = \langle a^2\rangle$ and $\Phi(G) = \langle t, a^2\rangle$, where a^2 is of order ≥ 4. It follows from the choice of x that $\Phi(G) = \langle a^2, x^2\rangle$. We have $t^x = tu$ since $T = C_G(t)$. If x would centralize a^2, then x centralizes $\langle a^2, x^2\rangle = \Phi(G)$ and $\Phi(G)$ contains t so $x \in C_G(t) = T$, which is not the case. Also $C_G(x^2) > \langle T, x\rangle = G$, so $x^2 \in Z(G)$. Since $t \notin Z(G)$, so $Z(G) \le T$ is cyclic. We claim that $\langle x^2\rangle = Z(G)$. If not, then there is an element $y \in Z(G)$ with $y^2 = x^2$. But $y^2 \in \mho_1(C) = \Phi(T)$ and so $x^2 \in \mho_1(C)$ which is not the case. Moreover, if $y \in G\setminus T$, then $y^2 \in Z(G)$ since T is abelian and generates G together with y. As we saw, $a^2 \notin Z(G)$. All elements in $(G/Z(G))\backslash(T/Z(G))$ are involutions. Since $Z(G) \not\leq \Phi(T)$, it follows that $T/Z(G)$ is cyclic. In that case, $G/Z(G)$ is dihedral. Indeed, since $\Phi(G) \not\leq Z(G)$, we conclude that $G/Z(G)$ is not abelian of type (2,2). We have obtained a group in part (c) of the theorem.

In the rest of the proof we assume that G possesses a normal four-subgroup U which does not contain our involution t. Set again $T = C_G(U)$ and so t is not in T so that $G = \langle t\rangle T$. By Dedekind law, $G_0 = C_G(t) = \langle t\rangle \times A$, where $A = C_T(t)$ is cyclic of order 2^m, $m \geq 2$, and so $Z = \Omega_1(A) = C_U(t) = \Omega_1(Z(G))$. Since $Z(G) \subseteq A$, so $Z(G)$ is cyclic. Set $G_1 = N_G(G_0)$. Since G is nonabelian we have $G_1 \neq G_0$. Since $\Omega_1(G_0) = \langle t, z\rangle$ where $\langle z\rangle = Z$, t has only two conjugates, t and tz, in G_1. It follows that $|G_1 : G_0| = 2$ and so $G_1 = G_0U$ because $|G_0U : G_0| = 2$. Set $D_0 = U\langle t\rangle$ so that D_0 is a dihedral group of order 8 and G_1 is the central product $G_1 = A * D_0$ with $A \cap D_0 = \langle z\rangle$. We have $U = \langle u, z\rangle \subset D_0$. Let v be an element of order 4 in D_0 and let y be an element of order 4 in A so that we have $y^2 = v^2 = z$ and so $x = yv$ is an involution in $G_1 \setminus D_0$. Since $x^t = xz$, we see that $D_1 = \langle x, t\rangle$ is a dihedral group of order 8. Because $A = Z(G_1)$, we also have $G_1 = A * D_1$, $D_1 \cong D_8$, $t \in D_1$, and $D_1 \cap U = Z(D_1) = \langle z\rangle$.

In the rest of the proof we consider a subgroup S of G which is maximal subject to the following conditions:

(i) S contains $G_1 = A * D_1$,

(ii) $S = AL$, where L is a normal subgroup of S and $L \cong D_{2^n}$, $n \geq 3$,

(iii) $A \cap L = \langle z\rangle = Z(L)$ and

(iv) $L \cap G_1 = D_1 \cong D_8$.

We note that $A = \langle a\rangle$ is cyclic of order 2^m, $m \geq 2$, and $\Omega_1(A) = \langle z\rangle = Z(D_1) = Z(L) = \Omega_1(Z(G))$. We have $t \in D_1$, $C_G(t) = \langle t\rangle \times A$ and $A = Z(G_1)$. Hence $Z(G) \subseteq A$ and so $Z(G)$ is cyclic of order $\leq 2^m$. Also, G_1 contains $U = \langle z, u\rangle$ which is a normal four-subgroup of G and $u^t = uz$. Now we act with $A = \langle a\rangle$ on the dihedral group L, where we set $L = \langle b, t \mid b^{2^{n-1}} = 1, t^2 = 1, b^t = b^{-1}\rangle$. Here $\langle b\rangle$ is the unique cyclic subgroup of index 2 in L and so $\langle b\rangle$ is A-admissible. We have $|\langle b\rangle \cap D_1| \geq 4$, A centralizes D_1 and $C_L(a) = \langle t\rangle(C(a) \cap \langle b\rangle)$ and so $C_L(a)$ is a dihedral subgroup of L of order ≥ 8 containing D_1. Looking at $\mathrm{Aut}(L)$ (Proposition 2.6) we see that A induces on L a cyclic group of automorphisms of order at most 2^{n-3} and so $|A : C_A(L)| \leq 2^{n-3}$. Since S/L is cyclic, we have $\Omega_1(S) = UL$. Set $\Omega_1(S) \cap A = \langle y\rangle$, where y is an element of order 4 with $y^2 = z$ and we have $\langle y\rangle = \Omega_2(A)$. Let us consider at first the case that y acts faithfully on L so that $C_A(L) = \langle z\rangle$ and $\langle a\rangle/\langle z\rangle$ acts faithfully on L. Then y induces an automorphism of order 2 on L and since $C_L(a) \supseteq D_1$ we must have $n \geq 4$, $[y, t] = 1$, and $b^y = bz$ so that $C_L(y) = \langle t, b^2\rangle \cong D_{2^{n-1}}$ and $C_L(y)$ is a maximal subgroup of L. Let v be an element of order 4 in D_1. Since

$U = \langle z, u\rangle \subseteq AD_1$, we may set $u = yv$. We have $y^b = yz = y^{-1}$ and so $u^{tb} = (uz)^b = (yvz)^b = y^{-1}vz = yv = u$ so that $\langle z, u, tb\rangle$ is an elementary abelian subgroup of order 8. Obviously, $C(u) \cap \Omega_1(S) = \langle u\rangle \times \langle tb, b^2\rangle$ and since $C_G(t) = A \times \langle t\rangle$ does not contain an E_8, t cannot be fused in G to any involution contained in $C(u) \cap \Omega_1(S)$. Naturally, t cannot be fused in G to any involution in U because U is normal in G. On the other hand, it is easy to compute that any involution in $\Omega_1(S)$ lies either in L or in $C(u) \cap \Omega_1(S)$. This forces $N_G(S) = S$ and so $S = G$. If $m = 2$, we have the Berkovich groups for $m = 2$ stated in (g) of Theorem 1.1 since $\langle t, b^2\rangle \cong D_{2^{n-1}}$ centralizes A and the involution tb acts invertingly on A. So assume that $m \geq 3$. Then we may set (see Proposition 2.6) $b^a = b^{1+2^i}$, where $i \geq 2$ because $C_L(a) \supseteq D_1$. Since $C_L(a) \cong D_{2^{i+1}}$ and $C_L(a^{2^{m-1}}) \cong D_{2^{i+m}}$ and $a^{2^{m-1}} = z$, $C_L(a^{2^{m-1}}) = L$. It follows that $i = n - m$. We have obtained exactly the groups stated in part (e) of Theorem 1.1.

In what follows we shall assume always that $\langle y\rangle = \Omega_2(A)$ centralizes L so that $\Omega_1(S)$ is the central product of $\Omega_2(A)$ and L. In this case each involution in $\Omega_1(S)$ is contained either in L or in U and so the subgroup L being generated by its own involutions is a characteristic subgroup of $\Omega_1(S)$ and so of S. If $S = G$, then we get some groups stated in part (d) of Theorem 1.1

From now on we shall assume in addition that $S \neq G$. Let $W = N_G(\Omega_1(S))$. Since W fuses $ccl(t) \cap \Omega_1(S)$ with $ccl(tb) \cap \Omega_1(S)$, where both classes are of size 2^{n-2} and both are contained in L, we get $|W : S| = 2$, S is normal in W, so L is normal in W. If $\Omega_1(S) = \Omega_1(W)$, then we have $W = G$. Suppose that $W \neq G$ and let $g \in N_G(W) \setminus W$ so that $g^2 \in W$. We have $\langle ccl(t) \cap W\rangle = L$, U is normal in G, and all involutions in $\Omega_1(S)$ lie in U or in L. Therefore (replacing g with gw, $w \in W$, if necessary) we may assume that $s = t^g \in W \setminus S$. Since s normalizes L and $C(t) \cap (L\langle s\rangle) = \langle t, z\rangle$, we have $L\langle s\rangle \cong D_{2^{n+1}}$. Now L is normal in W, L^g is normal in W, $L^g \cap S = L^g \cap L$, $L^g S = W$, $|L^g S : S| = |L^g : (L^g \cap S)| = |L^g : (L^g \cap L)| = 2$, and $|LL^g| = 2^{n+1}$ so $LL^g = L\langle s\rangle$. Hence $(LL^g)^g = LL^g$ and so $L\langle s\rangle$ is a dihedral group of order 2^{n+1} which is normal in W. We have $W = A(L\langle s\rangle)$ and this contradicts the maximality of $S = AL$. Hence we must have $W = G$ in any case. If $\Omega_1(S) = \Omega_1(G)$ then we get groups stated in part (d) of Theorem 1.1.

In what follows we assume in addition that $\Omega_1(S) \neq \Omega_1(G)$. Then for each involution $s \in G \setminus S$, we have that s normalizes L and so $M = L\langle s\rangle \cong D_{2^{n+1}}$ (since $C(t) \cap M = \langle t, z\rangle$). Because of the maximality of S, we have that M is not normal in G. But L is normal in G and so $A_0 = C_G(L) = C_A(L)$ (containing $\Omega_2(A)$) is also normal in G and we have $A_0 = Z(S)$, which is of order ≥ 4. We look at the structure of $\bar{G} = G/L$ which is a group with a cyclic subgroup $\bar{S} = \bar{A} = (AL)/L \cong A/\langle z\rangle$ (bar convention !) of order 2^{m-1} and index 2 and with an involution $\bar{M} = M/L$ outside of the cyclic group. Since M is not normal in G, $\bar{G}$ is nonabelian. We have $\bar{G} = \bar{A}\bar{M}$, $\bar{A} \cap \bar{M} = 1$, and $|\bar{A}| \geq 4$, so that $m \geq 3$. Now $|A_0| \geq 4$ and so $|\bar{A}_0| \geq 2$ and $\bar{A}_0$ is normal in $\bar{G}$ and $\bar{A}_0 \subseteq \bar{A}$. Since $\bar{G}/\bar{A}_0$ is a subgroup of the outer automorphism group of the dihedral group L, it follows that $\bar{G}/\bar{A}_0$ is abelian.

Suppose at first that $\bar{G}$ is not of maximal class so that $\Omega_1(\bar{G}) = (UM)/L$ and $UM = \Omega_1(G)$. Since M is not normal in G, it follows that s must act invertingly on $Z(\Omega_1(S)) = \Omega_2(A)$ and so $\langle z, u, s\rangle \cong E_8$. Namely, if s centralizes $\Omega_2(A)$, then $\Omega_1(G) = \Omega_2(A) * M$ and each involution in $\Omega_1(G)$ lies in U or in M and so M would be normal in G, which contradicts the maximality of S. If we set $A = \langle a\rangle$, then $N_G(M) = M\langle a^2\rangle$, which follows from the structure of $\bar{G}$ (see Propositions 2.2 and 2.3). Since $\Omega_2(A)$ centralizes L but $\Omega_2(A)/\langle z\rangle$ acts faithfully on M, it follows that $\langle a^2\rangle/\langle z\rangle$ acts faithfully on M and so $\langle a^2\rangle/\Omega_2(A)$ acts faithfully on L and $C_G(L) = \Omega_2(A) = A_0$. It follows that $Z(G)$ is of order 2 and $A/\Omega_2(A)$ acts faithfully on L. Because $\bar{s}$ acts faithfully on $\bar{A}$, $|\bar{A}| \geq 8$ and $m \geq 4$. We may assume that the involution s acts on $L = \langle b, t\rangle$ as follows: $b^s = b^{-1}$ and $t^s = tb$ and we set $b^{2^{n-2}} = z$, where $\langle z\rangle = Z(G)$. Replacing a with a^r (r odd) we may assume $b^a = b^{1+2^i}, i \geq 2$ because $C_L(a) \supseteq D_1 \cong D_8$. Hence we have $C_L(a) \cong D_{2^{i+1}}$ and so $C_L(a^{2^{m-2}}) \cong D_{2^{i+m-1}} \cong L \cong D_{2^n}$, where we have taken into account that $\langle a\rangle/\langle a^{2^{m-2}}\rangle = A/\Omega_2(A)$ acts faithfully on L and $\Omega_2(A)$ centralizes L. Thus $i + m - 1 = n$ and so $i = n - m + 1$. Since $i \geq 2$ we must have $n \geq m + 1$. Because $m \geq 4$, we have here $n \geq 5$. It remains to determine the commutator $[a, s]$. We know that $A = \langle a\rangle$ does not normalize M and so $\langle[\bar{a}, \bar{s}]\rangle = \bar{A}_0$ which gives $[a, s] = a_0 l$, where $\langle a_0\rangle = A_0$ has order 4, $a_0^2 = z$ and $l \in L$. We can also write $[a, s] = (a_0 z)(zl) = a_0^{-1}(zl)$ and so replacing l with zl, if necessary, we may assume that $a_0 = a^{2^{m-2}}$ and s inverts $\langle a_0\rangle$. Hence we get $a^{-1}sas = a_0 l$ and so $a^{-1}sa = a_0 ls$. Since $a_0 ls$ is an involution, we must have $l = b^j$ for a suitable integer j. Therefore $t^{a^{-1}sa} = t^{a_0 b^j s}$ which gives $b^a = b^{1-2j}$. On the other hand $b^a = b^{1+2^i}$ and so $j \equiv -2^{i-1} \pmod{2^{n-2}}$. This gives (4) $a^{-1}sa = a^{2^{m-2}}b^{-2^{i-1}}s$ or (5) $a^{-1}sa = a^{2^{m-2}}b^{-2^{i-1}}sz$. However, replacing b with $b' = b^{1-2^{n-i-1}}$ and t with $t' = tb^{2^{n-i-2}}$, we see that all the relations obtained so far in this case go into the same relations with b' instead of b and t' instead of t but the relation (4) goes into relation (5). Therefore we may choose the relation (4). We have determined the structure of G uniquely in this case and this is the group given in (f) of Theorem 1.1.

Suppose now that $\bar{G} = G/L$ is a group of maximal class. Since $\bar{G}/\bar{G}'$ is of order 4 and $\bar{G}/\bar{A}_0$ is abelian, we have $|\bar{A} : \bar{A}_0| \leq 2$.

We consider at first the case $\bar{A} = \bar{A}_0$ which means that $A = C_G(L)$ and so S is the central product of A and L with $A \cap L = \langle z\rangle = Z(L)$. Hence $A = Z(S)$ and so A is normal in G. Since $(A\langle s\rangle)/\langle z\rangle$ is of maximal class and the case $[A, s] \subseteq \langle z\rangle$ is not possible (because $L\langle s\rangle$ is not normal in G), we have either $a^s = a^{-1}$ or $a^s = a^{-1}z$, which is possible only when $m \geq 3$. Our group G is isomorphic to a Berkovich group $F(m, n)$ as stated in part (g) of Theorem 1.1. Also we have here $\langle z, u, s\rangle \cong E_8$.

It remains to consider the case where $|\bar{A} : \bar{A}_0| = 2$ so that $C_G(L) = A_0$ is of index 2 in A. We have $A_0 = \langle a^2\rangle = Z(S)$ and so a induces an automorphism of order 2 on L with $C_L(a) \supseteq D_1$. This gives at once that $b^a = bz$ and so $n \geq 4$. Hence we also have $a^b = az$. Since $[A, L] = \langle z\rangle$ and $L/\langle z\rangle$ is a (nonabelian) dihedral group of order 2^{n-1}, it follows that $C_S(L/\langle z\rangle) = A * \langle b'\rangle$, where $\langle b'\rangle$ is the cyclic subgroup of order 4 in L. Hence $A * \langle b'\rangle$ is normal in G. If s normalizes A, then $a^s = a^{-1}z^\alpha$, $\alpha = 0, 1$ since $A\langle s\rangle/\langle z\rangle$ is of maximal class. But from $sts = tb$ we get acting on a : $a^{sts} = a^{tb}$ which gives $az = a$ and this is a contradiction. Hence s does not normalize A. Since $A^s \subseteq A * \langle b'\rangle$ and $(A\langle b'\rangle/\langle b'\rangle)\langle s\rangle$ is of maximal class, we may set (6) $a^s = a^{-1}b'$ or (7) $a^s = a^{-1+2^{m-2}}b'$. We must have $a^{s^2} = a$ since $s^2 = 1$. But in case (6) this gives $(a^{-1}b')^s = a$ and so $az = a$ which is a contradiction. Hence we must have the relation (7). Replacing a with a^{-1} in (7), we get $(a^{-1})^s = (a^{-1})^{-1+2^{m-2}}b'^{-1}$ and other relations in this case are not changed. Thus we may put $b' = b^{2^{n-3}}$ in (7). This determines the group G as stated in part (h) of Theorem 1.1. Here we have also $\langle z, u, s\rangle \cong E_8$.

An inspection of obtained groups shows that in any case G has no elementary abelian subgroups of order 16 and G is generated by at most 3 elements. Also in the case where G possesses an elementary abelian subgroup of order 8, we see that $Z(G)$ has order 2. Theorem 1.1 is proved.

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